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Boson–Fermion Correspondence on the Circle via Quantum Stochastic Calculus[†]

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The quantum stochastic differential formula $d\mathrm{B} = (-1)^{\Lambda}dA$, known to relate boson and fermion fields A and B, respectively, on the Fock space over $L^2(\mathbb{R}_+)$, is shown to hold in a modified form in a Fock space associated with the nontrivial complex line bundle over the circle S^1.

1. INTRODUCTION

One of the most striking applications of quantum stochastic calculus (Hudson *et al.* (1984); Parthasarathy (1992)) has been to relating boson and fermion fields. The quantum stochastic differential formula

$$d\mathrm{B} = (-1)^{\Lambda}dA \tag{1.1}$$

connects the Fock representation of the canonical commutation and anticommutation relations (CCR and CAR) over the test function space $L^2(\mathbb{R}_+)$ (Hudson *et al.* (1986)). Here A and B are the boson and fermion annihilation processes

$$A(t) = a(\chi_{[0,t]}), \qquad B(t) = b(\chi_{[0,t]}) \tag{1.2}$$

where $\chi_{[0,t]}$ denotes the indicator function of the interval [0, t] and Λ is the gauge or number process of quantum stochastic calculus; equivalently, $P = (-1)^{\Lambda}$ is the *parity process*, so that $P(t)$ $\mathbb{Z}_2$-grades the Fock space up to time t. Boson and fermion annhilation and creation fields can be constructed by the stochastic integral prescription

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.
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$$a(f) = \int \bar{f} dA, \qquad a^\dagger(f) = \int f dA^\dagger$$

$$b(f) = \int \bar{f} dB, \qquad b^\dagger(f) = \int f dB^\dagger f \in L^2(\mathbb{R}_+)$$

These satisfy the CCR and CAR

$$[a(f), a(g)] = 0, \qquad [a(f), a^\dagger(g)] = \langle f, g\rangle 1$$

$$\{b(f, b(g)\} = 0, \qquad \{b(f), b^\dagger(g)\} = \langle f, g\rangle 1$$

for arbitrary $f, g \in L^2(\mathbb{R}_+)$, and constitute the Fock representations of the CCR and CAR, respectively in so far as the vacuum vector is annihilated by both sets of annihilation operators and is cyclic for both sets of creation operators and the existence of a cyclic, annihilated vacuum characterizes both Fock representations up to unitary equivalence.

Boson–fermion unification schemes also arise in the theory of loop groups (Pressley *et al.* (1986)). Commonly the representations of the CAR which occur are not Fock representations, but are of the form

$$b_E(f) = b(Ef) + b^\dagger(\overline{E^\perp f})$$

where E is a certain projector acting on the test-function space (Carey *et al.* (1987); Ruijemaars (1989)). Moreover, the CCR representation which occurs is over a different Hilbert space from that of the CAR. It is tempting to speculate that such unification schemes are similarly describable in the language of quantum stochastic calculus and that the use of "kinks" or "blips" (Pressley *et al.* (1986)) may be thereby avoided in the boson-to-fermion transition. While we are not yet able to accomplish this in the non-Fock context of the existing literature, we shall show in this paper that it can be done for Fock representations.

In one sense this is surprising, since a program to use, in effect, circular time in stochastic calculus is seriously obstructed by the absence of a fully coherent notion of adaptedness.

In a second sense our result is less surprising, even heuristically inevitable. Regarding $\mathbb{R}_+$ as a covering space for the circle

$$S^1 = \mathbb{R}_+/2\pi\mathbb{N}$$

we may think probabilistically that, as time t increases from the initial value 0, our system does not "know" until t reaches the value 2π whether time is circular or linear, and therefore the formula (1.1) ought to be valid also in the circular case at least for $t < 2\pi$. However, when t reaches 2π, whereas naively we would expect (1.1) to generate the circular periodicity in circular time, it is evident that it does not do so; a factor $(-1)^{\Lambda(2\pi)}$ intrudes. Because the square of this parity factor is the identity, a second circuit of the circle corrects it. Thus the correct Fock boson–fermion unification scheme lives on a double covering of the circle and the appropriate language for its rigorous description is that of a Fock space of square-integrable sections of a line bundle over S^1. Once this bundle has been identified, the detailed working out is comparatively straightforward.

In what follows we identify the circle with the space of complex numbers of unit modulus:

$$S^1 = \{z \in \mathbb{C}: |z| = 1\}$$

The argument of $z \in S^1$ is the real number θ satisfying $0 \leq \theta < 2\pi$ such that $z = |z|\, e^{i\theta}$. Hilbert space inner products, denoted $\langle \cdot, \cdot \rangle$, are always linear on the right; all Hilbert spaces are complex.

2. A FOCK SPACE OF SQUARE-INTEGRABLE SECTIONS

We denote by $\mathfrak{h}$ the Hilbert space of measurable square-integrable functions $f: S^1 \to \mathbb{C}$ satisfying

$$f(z) = -f(-z), \qquad z \in S^1 \tag{2.1}$$

Though it is unnecessary for understanding what follows, we observe that $\mathfrak{h}$ comprises the square-integrable sections of the nontrivial complex line bundle over S^1 associated with the principal $\mathbb{Z}_2$-bundle ξ formed from the action of the group $\mathbb{Z}_2 = \{1, -1\}$ on S^1 in which the elements ∓ 1 act by multiplication. The base space $S^1/\mathbb{Z}_2$ is identified with S^1 in such a way that ξ becomes a double covering of S^1 by S^1 by identifying each point $\{z, -z\} \in S^1/\mathbb{Z}_2$ with $z^2 \in S^1$. If $\mathbb{Z}_2$ acts on $\mathbb{C}$, also by multiplication, we obtain the associated vector bundle $\xi[\mathbb{C}]$, of which the square-integrable sections are identified with functions $f: S^1 \to \mathbb{C}$ satisfying (2.1) by (Isham (1989), theorems, p. 149). This bundle-theoretic viewpoint becomes important for multidimensional generalizations.

For $t \in [0, 2\pi)$ we denote by

$$\mathfrak{h} = \mathfrak{h}_t \oplus \mathfrak{h}^t \tag{2.2}$$

the canonical decomposition of $\mathfrak{h}$ into the direct sum of the subspaces $\mathfrak{h}_t$ and $\mathfrak{h}^t$ comprising elements which vanish on the arcs

$$\{z \in S^1: 0 \leq \arg z < \tfrac{1}{2}t\} \qquad \text{and} \qquad \{z \in S^1: \tfrac{1}{2}t \leq \arg z < \pi\}$$

respectively [and consequently also on

$$\{z \in S^1 \colon \pi \le \arg z < \pi + \tfrac{1}{2}t\} \qquad \text{and}$$
$$\{z \in S^1 \colon \pi + \tfrac{1}{2}t \le \arg z < 2\pi\}$$

respectively, in view of (2.1)]. We denote by χ_t the element of $\mathfrak{h}$

$$\chi_t = \begin{cases} 1, & 0 \le \arg z < \frac{1}{2}t \\ -1, & \pi \le \arg z < \pi + \frac{1}{2}t, \quad t \in [0, 2\pi) \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

and by p_t the self-adjoint unitary operator of multiplication by $(-1)^{\chi_t}$. Note that this does indeed map $\mathfrak{h}$ to itself.

Now let $\mathcal{F}(\mathfrak{h})$ be the Fock space over $\mathfrak{h}$, which is conveniently defined as the closed linear span of the exponential vectors $e(f)$, $f \in \mathfrak{h}$, satisfying

$$\langle e(f), e(g)\rangle_{\mathcal{H}} = \exp\langle f, g\rangle_{\mathfrak{h}}$$

Corresponding to the decomposition (2.2) we may write

$$\mathcal{F}(\mathfrak{h}) = \mathcal{F}(\mathfrak{h}_t) \otimes \mathcal{F}(\mathfrak{h}^t) \tag{2.4}$$

where $\mathcal{F}(\mathfrak{h}_t)$ and $\mathcal{F}(\mathfrak{h}^t)$ are the Fock spaces over $\mathfrak{h}_t$ and $\mathfrak{h}^t$, respectively, and

$$e(f) = e(f_t) \otimes e(f^t)$$

where $f = (f_t, f^t)$ in the decomposition (2.2). The Fock annihilation operators $A_S(t) = a(\chi_t)$ are defined by the actions

$$A_S(t)e(f) = \langle \chi_t, f\rangle\, e(f) = 2\int_0^{t/2} f(e^{i\theta})\, d\theta\; e(f) \tag{2.5}$$

Together with their adjoints, which act on the exponential domain as

$$A_S^\dagger(t)e(f) = \frac{d}{d\epsilon}\, e(f + \epsilon\chi_t)\Big|_{\epsilon=0}$$

they form the *circular Boson annihilation and creation processes*

$$A_S = (A_S(t),\ t \in [0, 2\pi)), \qquad A_S^\dagger = (A_S^\dagger(t),\ t \in [0, 2\pi))$$

The *circular parity process* P_S consists of the self-adjoint unitary second quantizations of the operators p_t, thus $P_S(t)$ acts on exponential vectors as

$$P_S(t)e(f) = e(p_t f)$$

3. AN ISOMETRY

Consider the map $w\colon \mathfrak{h} \to L^2(\mathbb{R}_+)$ given by

$$(wf)(t) = \begin{cases} f(e^{it/2}), & 0 \le t < 2\pi \\ 0, & 2\pi \le t < \infty \end{cases}$$

w is isometric since

$$\|wf\|^2_{L^2(\mathbb{R}_+)} = \int_0^{2\pi} |f(e^{it/2})|^2 \, dt$$
$$= 2\int_0^{\pi} |f(e^{is})|^2 \, ds$$
$$= \|f\|^2_{\mathfrak{h}}$$

Hence its second quantization W, defined on exponential vectors by

$$We(f = e(wf), \qquad f \in \mathfrak{h}$$

is an isometry from $\mathcal{H}$ to the Fock space $\mathcal{F}(L^2(\mathbb{R}_+))$ over $L^2(\mathbb{R}_+)$. The adjoint W^* of W is similarly the second quantization of the co-isometry w^*: $L^2(\mathbb{R}_+) \to \mathfrak{h}$ defined by

$$w^*f(e^{i\theta}) = \begin{cases} f(2\theta), & 0 \leq \theta < \pi \\ -f(2(\theta - \pi)), & \pi \leq \theta < 2\pi \end{cases}$$

The usual annihilation, creation, and number processes A, $A^\dagger$, and Λ are defined in $\mathcal{F}(L^2(\mathbb{R}_+))$ by their actions on exponential vectors

$$A(t)e(f) = \int_0^t f(s) \, ds \, e(f)$$
$$A^\dagger(t)e(f) = \frac{d}{d\epsilon} e(f + \epsilon\chi_{[0,t[})\Big|_{\epsilon=0}$$
$$\Lambda(t)e(f) = \frac{d}{d\epsilon} e(e^{\epsilon\chi|[0,t]}f)\Big|_{\epsilon=0}$$

Here $\chi_{[0,t]}$ is the indicator function of the interval $[0, t]$.

Theorem 3.1. For $0 \leq t < 2\pi$

$$W^*A(t)W = A_S(t) \tag{3.1}$$
$$W^*A^\dagger(t)W = A_S^\dagger(t) \tag{3.2}$$
$$W^*(-1)^{\Lambda(t)}W = P_S(t) \tag{3.3}$$

Proof. We verify (3.1) by checking actions on exponential vectors, thus, for $f \in \mathfrak{h}$

$$W^*A(t)We(f) = W^* \int_0^t wf(s) \, ds \, e(wf)$$
$$= \int_0^t f(e^{is/2}) \, ds \, e(f)$$

$$= 2 \int_0^{t/2} f(e^{i\theta})\, d\theta\; e(f)$$
$$= A_S(t)e(f)$$

by (2.5). Evidently (3.2) follows from (3.1) by taking adjoints. For (3.3) we have

$$\begin{aligned} W^*(-1)^{\Lambda(t)}We(f) &= W^*e((-1)^{\chi[0,t]}wf) \\ &= e(p_t f) \\ &= P_S(t)e(f). \quad \blacksquare \end{aligned}$$

4. BOSON–FERMION UNIFICATION ON THE CIRCLE

The stochastic integral prescription

$$b(f) = \int \overline{f(s)}(-1)^{\Lambda(s)}\, dA(s)$$

$$b^{\dagger}(f) = \int f(s)(-1)^{\Lambda(s)}\, dA^{\dagger}(s)$$

generates the Fock representation of the CAR over $L^2(\mathbb{R}_+)$ (Hudson *et al.* (1986)). In particular, the *Fermion annihilation* and *creation processes*

$$B = (b(\chi_{[0,t]}),\, t \in \mathbb{R}_+), \qquad B^{\dagger} = (b^{\dagger}(\chi_{[0,t]}),\, t \in \mathbb{R}_+)$$

satisfy

$$dB = (-1)^{\Lambda}\, dA, \qquad dB^{\dagger} = (-1)^{\Lambda}\, dA^{\dagger} \tag{4.1}$$

We use the isometry W of Section 3 to transfer this structure to the Fock space $\mathcal{F}(\mathfrak{h})$.

Theorem 4.1. The operators defined for $f \in \mathfrak{h}$ by

$$b_S(f) = W^*b(wf)W, \qquad b_S^{\dagger}(f) = W^*b^{\dagger}(wf)W$$

together with the vacuum vector $e(0)$, constitute the Fock representation of the CAR over $\mathfrak{h}$. That, is they satisfy

(a) $\{b_S(f), b_S(g)\} = 0, \qquad \{b_S(f), b_S^{\dagger}(g)\} = \langle f, g\rangle\, 1, \qquad f, g \in \mathfrak{h}$

(b) $b_S(f)e(0) = 0, \qquad f \in \mathfrak{h}$

(c) $\{b_S^{\dagger}(f_n) \ldots b_S^{\dagger}(f_1)e(0),\; n = 0, 1, \ldots, f_1, \ldots, f_n \in \mathfrak{h}\}$

is total in $\mathcal{F}(\mathfrak{h})$

Proof. We note first that, corresponding to the canonical factorization

$$\mathcal{F}(L^2(\mathbb{R}_+)) = \mathcal{F}(L^2[0, 2\pi)) \otimes \mathcal{F}(L^2[2\pi, \infty))$$

the operator WW^* takes the form

$$WW^* = 1 \otimes |e(0)\rangle\langle e(0)|$$

where $|e(0)\rangle\langle e(0)|$ is the projector onto the vacuum $e(0)$ in $\mathcal{F}(L^2[2\pi, \infty))$. Since w maps $\mathfrak{h}$ onto $L^2[0, 2\pi)$, the operator $b(wf)$ takes the form $b_{2\pi}(wf) \otimes 1$ in the same decomposition, and $g \mapsto b_{2\pi}(g)$ is the Fock representation of the CAR over $L^2[0, 2\pi]$. It follows that

$$\begin{aligned}
\{b_S(f), b_S(g)\} &= \{W^* b_{2\pi}(wf) \otimes 1W, W^* b_{2\pi} f(wg) \otimes 1W\} \\
&= W^*(\{b_{2\pi}(wf), b_{2\pi}(wg)\} \otimes |e(0)\rangle\langle e(0)|)W \\
&= 0
\end{aligned}$$

and similarly

$$\begin{aligned}
\{b_S(f), b_S^\dagger(g)\} &= W^*(\{b_{2\pi}(wf), b_{2\pi}^\dagger(wg)\} \otimes |e(0)\rangle\langle e(0)|)W \\
&= W^*(\langle wf, wg\rangle\, 1 \otimes |e(0)\rangle\langle e(0)|)W \\
&= \langle f, g\rangle\, W^*WW^*W \\
&= \langle f, g\rangle\, 1
\end{aligned}$$

so we have the CAR. Also,

$$\begin{aligned}
b_S(f)e(0) &= W^* b(wf) W e(0) \\
&= W^* b(wf) e(w0) \\
&= W^* b(wf) e(0) \\
&= 0
\end{aligned}$$

by the vacuum annihilation property of the Fock representation b over $L^2(\mathbb{R}_+)$. Finally, we have

$$\begin{aligned}
b_S^\dagger(f_n) \ldots b_S^\dagger(f_1) e(0) &= W^* b_{2\pi}^\dagger(wf_n) \otimes |e(0)\rangle\langle e(0)| WW^* b_{2\pi}(wf_{n-1}) \\
&\quad \otimes |e(0)\rangle\langle e(0)| W \\
&\quad \times \ldots W^* b_{2\pi}(wf_1) \otimes |e(0)\rangle\langle e(0)| We(0) \\
&= W^* b_{2\pi}^\dagger(wf_n) \ldots b_{2\pi}^\dagger(wf_1) e(0) \otimes e(0)
\end{aligned}$$

But the vacuum $e(0)$ in $\mathscr{F}(L^2[0, 2\pi])$ is cyclic for the representation $b_{2\pi}$ of the CAR over $L^2[0, 2\pi]$. Hence the vectors of the form

$$b_{2\pi}^\dagger(wf_n) \ldots b_{2\pi}^\dagger(wf_1)e(0) \otimes e(0), \qquad n = 0, 1, \ldots, f_1, \ldots, f_n \in \mathfrak{h}$$

are total in the range of W, on which W^* acts isometrically, thus transforming this total set into a total set in $\mathscr{F}(\mathfrak{h})$ as required. ■

Introducing the fermion annihilation and creation processes B_S and $B_S^\dagger$ defined by

$$B_S(t) = b_S(\chi_t), \qquad B_S^\dagger(t) = b_S^\dagger(\chi_t)$$

where χ_t is given by (2.3), we may write formally, in view of (3.3) and (4.1),

$$dB_S = P_S\, dA_S, \qquad dB_S^\dagger = P_S\, dA_S^\dagger$$

In fact, a theory of stochastic integrals against dA_S and $dA_S^\dagger$ as integrators can be constructed directly, in imitation of (Hudson *et al.* (1984)). However, it is more economical to derive the boson–fermion unification on the circle from that on the line.

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